Bootstrap 2018 School: Integrability Problem Sets 1
O(N) model in 1+1 dimensions

Exercises for the first lecture: The problems on the S-matrix of the O(N) model and a small problem on the relation between the conserved charges and the factorization. The exercises with the asterisks $*$ are the ones that are not so much related to the main contents of the lectures and I recommend you to skip them if you do not have much time (or you are not so motivated to work on them).

1 O(N) S-matrix

Yang-Baxter

As explained in the lecture, by imposing two different Yang-Baxter equations, one can fix the form of the S-matrix of the O(N) model to be

$$S_{ij}^{kl}(\theta) = \sigma(\theta)\left[\delta_i^k\delta_j^l + \frac{1}{c\theta}\delta_i^l\delta_j^k + \frac{1}{d-c\theta}\delta_{ij}\delta^{kl}\right]. \tag{1}$$

The goal of this exercise is to determine the unfixed coefficients c, d and $\sigma(\theta)$ by imposing the combination of the Yang-Baxter equation, crossing symmetry and the unitarity condition.

1. By considering an appropriate 3-to-3 scattering process and imposing the Yang-Baxter equation, show that $d = -\frac{N-2}{2}$.
(*Hint*: In order to obtain a number that depends on N, one has to consider a scattering process which contains a closed loop of the $O(N)$ index.)

Now, after fixing d, one can impose the crossing symmetry and determine c as explained in the lecture. After doing so, the S-matrix is given by

$$S_{ij}^{kl}(\theta) = \sigma(\theta)\left[\delta_i^k\delta_j^l - \frac{2\pi\Delta}{\theta}\delta_i^l\delta_j^k + \frac{2\pi i\Delta}{\theta - i\pi}\delta_{ij}\delta^{kl}\right], \tag{2}$$

with $\Delta = 1/(N-2)$. The crossing symmetry also gives a constraint on $\sigma(\theta)$:

$$\sigma(\theta) = \sigma(i\pi - \theta). \tag{3}$$

A Different Viewpoint on the Crossing Symmetry

One can derive the crossing symmetry constraint on σ, $\sigma(\theta) = \sigma(i\pi-\theta)$, also from a different physical argument. The idea is to consider the following two-particle state made up of a particle and an anti-particle,

$$(\mathsf{Singlet}) \equiv \sum_i \phi^i(\theta + i\pi)\otimes\phi^i(\theta), \tag{4}$$

which we call the *singlet pair*, and scatter it against the third particle $\phi^j(\theta')$.

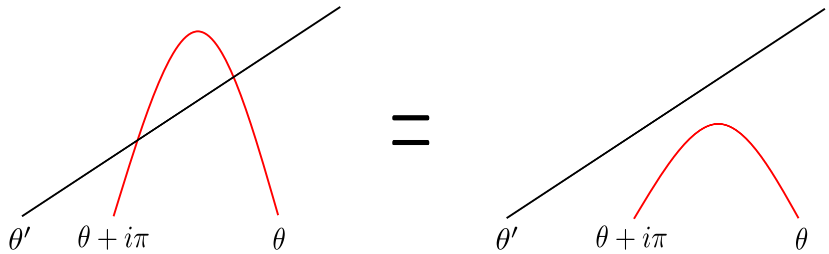

Figure 1: Pictorial representation of the decoupling of the singlet. The red line denotes the singlet pair and the black line denotes the third particle. The fact that the red lines for the singlet pair are connected indicates that we are summing over the indices.

2*. Check that the singlet pair has zero total energy and momentum. (It is also a singlet of $\mathrm{O}(N)$ since the indices are contracted.)

Since the singlet state has exactly the same quantum number as the vacuum, an arbitrary number of singlet states can be produced by the pair creation process from the vacuum. (This phenomenon is often called the vacuum polarization.) In order to have a well-defined S-matrix, we should require that such singlet pairs do not affect the propagation of other physical particles (see figure **??**). In other words, the scattering of the singlet state and the third particle $\phi^j(\theta')$ must be trivial.

3*. Compute the scattering[1] between the singlet state and the third particle $\phi^j(\theta')$ and require that it is just an identity matrix. As a result, you will get

$$\sigma(\theta)\sigma(\theta - i\pi) = \frac{(\theta - i\pi)^2}{(\theta - i\pi)^2 + 4\pi^2\Delta^2}\,, \tag{5}$$

which can be obtained combining the crossing condition (3) and the unitarity condition (7).

This derivation is more suited for the spin-chain S-matrix describing the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at finite coupling as will be explained in the lectures (by me and also by Pedro).

Unitarity

The next step is to impose the unitarity and solve for $\sigma(\theta)$.

4. Impose the unitarity condition[2]

$$S(-\theta)S(\theta) = \mathbf{1}, \tag{6}$$

[1] Owing to the factorizability, it is essentially given by a product of two 2-2 S-matrix.
[2] One should interpret $S(-\theta)S(\theta)$ as the matrix multiplication and $\mathbf{1}$ is the unit matrix.

and show that it gives the following condition on σ:

$$\sigma(-\theta)\sigma(\theta) = \frac{\theta^2}{\theta^2 + 4\pi^2\Delta^2} \,. \tag{7}$$

(*Hint*: It is simpler to work with the pictorial notation introduced in the lecture rather than working with indices.)

Solving the equations for $\sigma(\theta)$

Now we are going to solve the equations for $\sigma(\theta)$, (3) and (7). Before doing this, it is useful to decompose the S-matrix into different representations.

5. (Easy) Figure out which representations of $\mathrm{O}(N)$ show up as the intermediate states in the 2-2 scattering process that we have been discussing.

6. Project the s-channel S-matrix ($ij \to kl$) to the symmetric-traceless representation, and rewrite the equations for σ in terms of the projected S-matrix (which we denote by s in what follows). Show that they are equivalent to the following two equations:

$$s(\theta)s(-\theta) = 1 \,, \tag{8}$$

$$s(\theta^+)s(\theta^-) = \frac{(\theta^+ - 2\pi i\Delta)\theta^-}{(\theta^- + 2\pi i\Delta)\theta^+} \tag{9}$$

Here the notation $\theta^\pm$ means $\theta^\pm \equiv \theta \pm i\pi/2$.

We now try to solve (8) and (9).

7. Take the logarithm of the equation (9) and Fourier-transform both sides. Once you do it, you will find that the equation can be easily solved in the Fourier space. Find a solution and transform it back to the θ space.

The solution you will get by doing the exercise 5 is not unique: From any solution to (8) and (9), one can always construct another solution by multiplying to it a factor which satisfies

$$\begin{aligned} f(\theta)f(-\theta) &= 1 \,, \\ f(\theta^+)f(\theta^-) &= 1 \,. \end{aligned} \tag{10}$$

8. Check that the following product satisfies the equations for $f(\theta)$:

$$f(\theta) = \prod_{k=1}^{M} \frac{\sinh\theta - i\cos\alpha_k}{\sinh\theta + i\cos\alpha_k} \,. \tag{11}$$

The factors on the right hand side of (11) are called the CDD (Castillejo-Dalitz-Dyson) factors. By multiplying them, one can change the analyticity of the S-matrix (namely one can introduce or remove poles in the S-matrix) without affecting the unitarity and the crossing.

9. Analyze the analytic properties of $s(\theta)$ you obtained and try to remove the pole in the physical strip ($\mathrm{Im}\,\theta \in [0,\pi]$) by multiplying the minimal number of the CDD factor.

In the end, you should get

$$s(\theta) = -\frac{\Gamma\left(1+\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}-\frac{i\theta}{2\pi}\right)\Gamma\left(\Delta-\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}+\Delta+\frac{i\theta}{2\pi}\right)}{\Gamma\left(1-\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}+\frac{i\theta}{2\pi}\right)\Gamma\left(\Delta+\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}+\Delta-\frac{i\theta}{2\pi}\right)}\,, \tag{12}$$

which is equivalent to the expression for $\sigma(\theta)$ shown in the lecture[3].

Large N analysis

We now test the S-matrix that we derived from the bootstrap against the perturbative large N analysis. For that purpose, let us briefly review the large N analysis explained in the lecture. The key idea is to express the partition function of the $O(N)$ sigma model by introducing the Lagrange multiplier σ as

$$\begin{aligned} Z &= \int D\phi D\sigma\; e^{-S[\phi,\sigma]}\,, \\ S[\phi,\sigma] &= \frac{1}{\kappa^2}\int d^2x\left[(\partial\phi^i)^2 + \sigma((\phi^i)^2-1)\right]\,, \end{aligned} \tag{13}$$

and integrate out the ϕ^i fields. After doing so, one obtains the path-integral of σ

$$\begin{aligned} Z &= \int D\sigma\, e^{-S_{\text{eff}}[\sigma]}\,, \\ S_{\text{eff}}[\sigma] &= -\left(\frac{1}{\kappa^2}\int d^2x\,\sigma\right) + \frac{N}{2}\log\det(-\partial^2+\sigma)\,. \end{aligned} \tag{14}$$

In the 't Hooft limit,

$$N\to\infty\,, \qquad \lambda \equiv \frac{\kappa^2 N}{2} : \text{fixed}\,, \tag{15}$$

The effective action becomes proportional to N and the σ path integral localizes to the saddle point.

10*. Repeat the same analysis in the presence of insertion of four scalar fields

$$\langle \phi^i(x_1)\phi^j(x_2)\phi^k(x_3)\phi^l(x_4)\rangle = \frac{1}{Z}\int D\phi D\sigma\;\phi^i(x_1)\phi^j(x_2)\phi^k(x_3)\phi^l(x_4)e^{-S[\phi,\sigma]}\,, \tag{16}$$

and perform the LSZ reduction to derive the S-matrix of four fundamental scalars[4].

11*. Compare what you got in the exercise 8 with the Large N expansion of the exact S-matrix that we obtained from the bootstrap, and confirm that they match.

[3] The relation between $\sigma(\theta)$ and $s(\theta)$ is $s(\theta) = \sigma(\theta)(\theta - 2\pi i\Delta)/\theta$.

[4] *Hint*: Also here, the first step is to integrate out ϕ's. Since the path integral is Gaussian, it produces the Wick contraction (with the effective propagator involving σ). One can then analyze the path integral of σ around the saddle point.

2 Boundary Integrability and Conserved Charges

Here we analyze the relation between the conserved charges and the factorizability of the S-matrix in the presence of the boundary. As in the lecture, we consider a non-relativistic field theory in $1+1$ dimensions, but now in the presence of a boundary.

12*. Consider the process in which two particles are scattered off the boundary. Repeat the analysis performed in the lecture and discuss the physical consequence of the existence of the infinitely many conserved charges.
(*Hint*: You frst need to figure out what the reasonable sets of conserved charges are, since even the momentum is *not* conserved in the presence of the boundary.)