Bootstrap 2018 School: Integrability Problem Sets 2
Harmonic Oscillator and Bethe Ansatz

The goal of these exercises is to try to relate the Bethe equation, which is a simple yet mysterious algebraic equation, to a more familiar object, the harmonic oscillator with the aim of getting more intuition into the physical meaning of the Bethe equation.

1 Bethe equation for the Harmonic Oscillator

In the lecture, we discussed the Bethe equation

$$e^{ip_k L}\prod_{j\neq k} S(p_k,p_j)=1\,, \tag{1}$$

which, in terms of the rapidity variables,

$$e^{ip}=\frac{u+\frac{i}{2}}{u-\frac{i}{2}}\,,\qquad S(p_1,p_2)=\frac{u_1-u_2-i}{u_1-u_2+i}\,, \tag{2}$$

becomes a simple algebraic equation

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L=\prod_{j\neq k}^{M}\frac{u_k-u_j+i}{u_k-u_j-i}\,. \tag{3}$$

Before analyzing this Bethe equation, let us analyze the standard harmonic oscillator in a non-standard way and see that the spectrum is also governed by a simple algebraic equation. The starting point of the analysis is the Schrodinger equation

$$-\frac{\hbar^2}{2m}\psi''(x)+\frac{m\omega^2x^2}{2}\psi(x)=E\psi(x)\,. \tag{4}$$

This can be rewritten in terms of the *quasi-momentum*

$$p(x)\equiv\frac{\hbar}{i}\frac{\psi'(x)}{\psi(x)} \tag{5}$$

as

$$p^2-i\hbar p'=2m\left(E-\frac{m\omega^2x^2}{2}\right)\,. \tag{6}$$

What we are going to do now is to come up with an ansatz for $p(x)$ and derive the constraints on the parameters that appear in the ansatz.

1. Analyze the equation (6) at large x, and show that $p(x)$ has the following large x expansion

$$p(x)=im\omega x+O(1/x)\,. \tag{7}$$

A natural ansatz for $p(x)$ which satisfies the above asymptotic property is

$$p(x) = im\omega x + \frac{\hbar}{i}\sum_{k=1}^{N}\frac{1}{x-x_k}\,. \tag{8}$$

We chose this ansatz because 1. We do not expect double, or higher poles in x since that would introduce an essential singularity to the wave function $\psi(x)$. 2. The residue of each simple pole is related to the order of zeros in the wave function $\psi(x)$ (Check this statement). Since we expect that the wave function contains only simple zeros (nodes), each residue must be $\hbar/i$.

2. Plug in the ansatz to the equation (6) and analyze it at large x to show that E and N are related by

$$E = \hbar\omega\left(N+\frac{1}{2}\right)\,. \tag{9}$$

3. Show that x_k's in the ansatz have to satisfy the following algebraic equations:

$$x_k = \frac{\hbar}{m\omega}\sum_{j\neq k}\frac{1}{x_k-x_j} \qquad (k=1,\ldots,N) \tag{10}$$

We know that the wave function for the harmonic oscillator is given by the Gaussian times the Hermite polynomials. Thus the equation (10) can be viewed as the equation for the zeros of the Hermite polynomials.

4*. (Just for fun. Completely optional.) Take any orthogonal polynomials that you like and try to derive the "Bethe equation" for the zeros of the polynomials.

2 Scaling limit of the SU(2) Bethe equation

As explained in the lecture, the Bethe equation for the SU(2) spin chain takes the following form

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L = \prod_{j\neq k}^{M}\frac{u_k-u_j+i}{u_k-u_j-i}\,. \tag{11}$$

To analyze this equation, let us take the logarithm and express it as

$$iL\log\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}} + 2\pi n_k = \sum_{j\neq k}^{M} i\log\frac{u_k-u_j+i}{u_k-u_j-i}\,. \tag{12}$$

Here the integers n_k represent the $2\pi i$ ambiguity of the logarithm and we call them the *mode numbers*. We then analyze the solutions to the equation in the following scaling limit:

$$L\to\infty\,,\quad u\sim L\,,\quad M:\text{fixed}\,. \tag{13}$$

At the leading order, we can drop the right hand side of the equation and get

$$u_k \simeq \frac{L}{2\pi n_k}\,. \tag{14}$$

This means that the roots with the same mode number are close to each other while the roots with different mode numbers are far separated.

5. Derive[1] the equation which governs the next leading correction[2]. You will find the equation which is equivalent to (10) up to some rescaling.

6. Solve numerically the equation that you got in the previous exercise. Use it to compute the first nonzero correction to the energy

$$E = \sum_{k=1}^{M} \frac{1}{u_k^2 + \frac{1}{4}}\,, \tag{16}$$

for a few small M. Do you see any patterns?

Let us now try to derive the pattern that you observed more analytically. For this purpose, it is useful to reformulate the Bethe equation in terms of the so-called Q-function defined in the following way

$$Q(u) \equiv \prod_{k=1}^{M}(u - u_k)\,. \tag{17}$$

6. Show that the Bethe equation is equivalent to the following equation on $Q(u)$:

$$T(u)Q(u) = \left(u - \frac{i}{2}\right)^L Q(u+i) + \left(u + \frac{i}{2}\right)^L Q(u-i)\,. \tag{18}$$

Here $T(u)$ is some (unknown) polynomial of degree L. This equation is called the Baxter equation and the function $Q(u)$ is called the Q-function.

7. Show that the energy of the spin chain is given by

$$E = i\partial_u \log \frac{Q(u+\frac{i}{2})}{Q(u-\frac{i}{2})}\,. \tag{19}$$

[1] *Hint 1*: The right hand side becomes relevant only if the mode numbers are equal and the roots are close to each other. It is therefore enough to consider the case where all the mode numbers are identical $n_k = n$.

[2] *Hint 2*: Assume that u_k admits the expansion

$$u_k = \frac{1}{2\pi n}\left(L + i\sqrt{2}z_k L^{\alpha} + \cdots\right)\,. \tag{15}$$

and determine the power $\alpha < 1$. After doing so, one should be able to derive the equation for z_k.

Q-functions and the Baxter equation play an important role also for solving $\mathcal{N} = 4$ SYM although this topic will not be covered in the lectures of this school: More precisely, there is a formalism called the quantum spectral curve (see [1]), which can be thought of as sophisticated generalization of the Baxter equation, and this formalism is considered as the best formalism for computing the spectrum of the single-trace operators in $\mathcal{N} = 4$ SYM at finite coupling.

8. Use the fact that the subleading correction to the Bethe roots in the scaling limit satisfies the same equation as the zeros of the Hermite polynomial, express the Q-function in the scaling limit in terms of the Hermite polynomial. After doing so, derive the pattern that you observed in exercise 5 from the asymptotic property of the Hermite polynomial (which you can derive for instance using the differential equation satisfied by the Hermite polynomials) and (19).

References

[1] N. Gromov, V. Kazakov, S. Leurent and D. Volin, "Quantum Spectral Curve for Planar $\mathcal{N} = 4$ Super-Yang-Mills Theory," Phys. Rev. Lett. **112**, no. 1, 011602 (2014) arXiv:1305.1939